Indian Statistical Institute, Bangalore Centre
B.Math. (III Year) : 2009-2010
Semester I : Semestral Examination
Introduction to Differential Equations

27.11.2009 Time : 3 hours Maximum Marks : 100

Note: The paper carries 106 marks. Any score above 100 will be treated as 100.

1. [12 marks] Let $S(t)$ = balance amount in a bank account at time t, with $S(0) = S_0$ being the initial deposit. Suppose that the bank pays interest that is compounded continuously at constant rate $r > 0$. Moreover assume that additional funds are continuously deposited into the account at constant rate $k \geq 0$. Assume that the rate of change of $S(\cdot)$ is equal to the net rate at which funds are added to the account. Formulate the initial value problem, and find the solution. Assuming S_0, r, k to be known, find the time T required for the original sum to double in value.

2. [13 + 4 marks] Consider the ODE

$$x''(t) + 4x'(t) + 4x(t) = \frac{1}{t^2 e^{2t}}, \quad t > 1.$$

(i) Find the general solution.

(ii) Find the solution $x(\cdot)$ such that $x(1) = 0, x'(1) = -\frac{1}{e^2}$.

3. [10 + 6 + 4 marks] (i) Find two linearly independent power series solutions to Hermite's ODE

$$x''(t) - 2tx'(t) + 2\lambda x(t) = 0, \quad t \in \mathbb{R},$$

where $\lambda \in \mathbb{R}$ is a fixed parameter.

(ii) Show that $x(\cdot)$ solves Hermite's ODE in (i) if and only if $y(t) = x(t)\exp(-\frac{1}{2}t^2)$, $t \in \mathbb{R}$ satisfies the ODE

$$y''(t) + (2\lambda + 1 - t^2)y(t) = 0, \quad t \in \mathbb{R}.$$

(iii) If λ is a nonnegative integer show that the ODE in (ii) has a solution $y(\cdot)$ such that $y(t) \to 0$ as $|t| \to \infty$.

4. [6 + 4 marks] (i) Let $\lambda \geq 0$. Let $x(\cdot)$ be a twice continuously differentiable function except possibly at $t = 0$. Set $v(t) = \sqrt{t}x(t), t > 0$. Show that $x(\cdot)$ is a solution to Bessel's ODE

$$t^2 x''(t) + tx'(t) + (t^2 - \lambda^2)x(t) = 0, \quad t > 0$$

if and only if $v(\cdot)$ satisfies

$$v''(t) + (1 - \frac{4\lambda^2 - 1}{4t^2})v(t) = 0, \quad t > 0.$$

(ii) Find the general solution to Bessel's ODE when $\lambda = \frac{1}{2}$.

5. [17 marks] Consider the initial value problem :

$$u_t(t,x) + bu_x(t,x) + cu(t,x) = f(t,x), \quad t > 0, x \in \mathbb{R}$$

with $u(0,x) = g(x), x \in \mathbb{R}$, where $b, c \in \mathbb{R}$ are constants, and f, g are continuously differentiable functions. Show that the unique solution is given by

$$u(t,x) = e^{-ct}g(x - tb) + \int_0^t e^{-c(t-r)} f(r, x + (r-t)b)dr,$$

for $t > 0, x \in \mathbb{R}$.

6. [8 + 7 marks] (i) Find the continuous function u which is harmonic inside the unit disc and satisfies $u(1,\phi) = \cos^3(\phi), -\pi \leq \phi < \pi$.

(ii) For $0 \leq r < 1, -\pi \leq \phi, \theta < \pi$ let

$$f(r,\phi;\theta) = \frac{1}{2\pi} \frac{(1-r^2)}{(1 - 2r\cos(\phi - \theta) + r^2)}$$

denote the Poisson kernel in the unit disc. Show that for any $r < 1, -\pi \leq \phi < \pi$

$$\int_{-\pi}^{\pi} f(r,\phi;\theta)d\theta = 1.$$

7. [15 marks] Find $(t,x) \mapsto u(t,x)$ which is bounded in x and satisfies the equation

$$4u_t(t,x) = u_{xx}(t,x), \quad t > 0, x \in \mathbb{R}$$

with the initial value $u(0,x) = \exp(2x - x^2), x \in \mathbb{R}$.